potential for nonzero V can be deduced from Fig. 3 of Ref. 1. This figure shows that the discharge is located downstream of the electrodes, in the freejet region. In the freejet region the pressure is lower than in the interelectrode region. Consequently, the breakdown voltage in this region is lower.

In general, prior to breakdown in noble gases, the electron drift velocity term $V_e/4D_e$ in Eq. (1) is much greater than the gas velocity term, $V^2/4D_e$.[2] Therefore, the breakdown electric field of gases in subsonic or supersonic MHD devices will generally be independent of gas convective effects.

Subsequent to breakdown, one obtains[2] an expression similar to Eq. (1) for the magnitude of the electric field necessary to sustain the discharge in the presence of transverse convective effects. However, the electron drift velocity term does not appear in the equation for the discharge-sustaining voltage. One, therefore, finds that for MHD devices the gas velocity term is generally much larger than the terms due to plasma transport phenomena. Thus, the discharge is generally controlled by gas convective effects. One usually obtains a discharge structure similar to that shown in Fig. 3, Ref. 1. Further details on this phenomena can be found in Ref. 4.

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Reply by Author to B. Zauderer

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THE explanation given by Zauderer to the reduction in the breakdown potential for a dynamic condition is certainly one possibility for describing such an effect. However, the work reported in Ref. 1 indicated that the onset of breakdown does not occur in the "U-shaped" form as shown in Fig. 3 of Ref. 2, but the onset of breakdown most likely occurred within the upstream quarter of the electrode surface. The photograph shown in Fig. 4 of Ref. 1 exhibits a vertical plasma column between and within the electrode surfaces. It should be noted that the discharge shown in Fig. 3 of Ref. 2 refers to a typical *continuously* operating discharge between flat parallel-plate electrodes with the presence of a gas flow and therefore does not necessarily represent the conditions present during the onset of breakdown.

The pressure differential between the electrode region and the freejet region was measured and found to be negligible. The uniform pressure is a consequence of the subsonic velocity and the fact that the region between the electrodes was open to the vacuum chamber on all sides. Therefore this would certainly not be compatible with the measured reduction in breakdown potential of 50–80% below that required for stagnation conditions.

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Comment on "Optimum Discrete Approximation of the Maxwell Distribution"

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IN Ref. 1, a set of discrete velocities and weighting coefficients is presented which optimizes a discrete approximation to the Maxwellian equilibrium distribution function. We would like to point out that this optimization has been obtained and reported some time ago and has been utilized in the solution of several rarefied gasdynanic flow problems. In Refs. 2–4, this was referred to as the "modified quadrature" or the "half-range quadrature." The same one-dimensional values for the discrete velocities and weighting coefficients which appear in Ref. 1 also appear in Ref. 5, which due to unfortunate delays was not printed until recently. However, the values were published much earlier in Ref. 6 for the same number of discretizations (up to 8 levels).

The derivation used in Ref. 1 is essentially that of Refs. 5 and 6 although the descriptive phrasing is somewhat different. We took the approach that

$$\int_0^\infty g(v_x)e^{-v_x^2}\,dv_x \cong \sum_{j=1}^{N} H_j\, g(\alpha_j) \tag{1}$$

where H_j are the weighting coefficients corresponding to the discrete velocities α_j and $g(v_x)$ is any function for which the integral is defined. The H_j and α_j are determined under the requirement that Eq. (1) is exact if $g(v_x)$ is a polynomial of degree $2N - 1$ or less. This corresponds to duplicating the first $2N$ moments of a normalized Maxwellian distribution. If g is not such a polynomial, then an error is incurred which depends upon the departure of g from this form. Equation (1) leads to the equation

$$\sum_{j=1}^{N} H_j\alpha_j{}^k = \int_0^\infty v_x{}^k e^{-v_x^2}\,dv_x,\ k = 0,1,\ldots,2N-1 \tag{2}$$

which is to be compared to Eq. (7) of Ref. 1

$$\sum_{i=1}^{N} K_i v_i{}^j = \frac{1}{(\pi)^{1/2}}\int_0^\infty v^j e^{-v^2}\,dv,\ j = 0,1,\ldots,2N-1$$

Thus, the one-dimensional K_i of Ref. 1 and the H_j of Ref. 5 differ by a factor of $1/(\pi)^{1/2}$ due to the normalizations used. The α_j and v_i are identical.

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The use of this type of discretizing of molecular velocities has much wider application than that of representing a Maxwellian distribution function. Using this discrete ordinate method in rarefied flow problems transforms the Boltzmann equation with the Bhatnagar-Gross-Krook model from an integro-differential equation to a system of differential equations and simplifies calculations considerably. This method has proven successful in cases where departures from equilibrium are small [in which the discretization under discussion here is very useful since the g function of Eq. (1) is a perturbation about the Maxwellian] and also for those where large departures from equilibrium occur[7–10] (in which this particular discretization is totally inadequate and different quadratures must be used).

There is one difference in the treatment of two- and three-dimensional distributions between Refs. 1 and 5. The authors of Ref. 1 are interested primarily in Maxwellian distributions and for higher dimensions use an energy variable rather than a velocity variable. However, for application of the discretization to nonequilibrium rarefied flow problems, particularly where a solid boundary is present, this cannot be done since one must distinguish between particles approaching a surface and those being emitted from a surface. This distinction is necessary in order to satisfy the boundary conditions for the problem. Thus, in gasdynamic problems the two- and three-dimensional discretization takes the form of a multiple usage of the one-dimensional results, i.e.,

$$\int_0^\infty \iint g(v_x,v_y,v_z)e^{-(v_x^2+v_y^2+v_z^2)}\, d^3v = \sum_{i=1}^{L}\sum_{j=1}^{M}\sum_{k=1}^{N} H_iH_jH_k g(\alpha_i,\alpha_j,\alpha_k) \quad (3)$$

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Reply by Authors to D. P. Giddens and A. B. Huang

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It is unfortunate that we had not noticed the results of Huang and Giddens[1] prior to the publication of our results.[2] However, we feel that we have made substantial contributions beyond those of Ref. 1. We have published numbers for the two- and three-dimensional cases which are not available in Ref. 1. Although these numbers may not be useful in the application mentioned in the comment, our treatment of this problem was in fact motivated by an application of them (Ref. 2, Sec. III).

We have treated the one-dimensional case in which there is an odd number of levels, as well as the two- and three-dimensional cases in which negative moments must be duplicated. Finally, we have published numbers for the one-dimensional case which differ substantially from those of Ref. 1 and which we believe to be more accurate. To support this assertion, we have calculated

$$\sum_{j=1}^{8} H_j\alpha_j^k \qquad k = (0, 1, \ldots, 15)$$

using the numbers in Ref. 1 for $n = 8$. According to the derivation of the roots and weights, these sums should give exact results for the corresponding moments. Yet errors ranging from 0.004% to 3.1% occur in the range $k = 8, 9, \ldots, 15$. In contrast, using the numbers from Ref. 2 gives results which are in error by less than one part in 10^6. In fact these numbers, prior to rounding off, recover these moments to one part in 10^{12} or better. We conclude that the numbers in Ref. 2 should be used in any calculations performed in preference to those of Ref. 1.

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Erratum: "Further Similar Laminar-Flow Solutions"

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Reference 5 of this paper should read: Rogers, D. F., "Reverse Flow Solutions for Compressible Laminar Boundary Layer Equations," *The Physics of Fluids*, Vol. 12, No. 3, March 1969, pp. 517–523.